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Defining relations of the noncommutative trace algebra of two 3×3 matrices

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To the anniversary of Amitai Regev—mathematician and friend

Abstract

The noncommutative (or mixed) trace algebra T_{nd} is generated by d generic $n \times n$ matrices and by the algebra C_{nd} generated by all traces of products of generic matrices, $n, d \geqslant 2$. It is known that over a field of characteristic 0 this algebra is a finitely generated free module over a polynomial subalgebra S of the center C_{nd}. For $n = 3$ and $d = 2$ we have found explicitly such a subalgebra S and a set of free generators of the S-module T_{32}. We give also a set of defining relations of T_{32} as an algebra and a Gröbner basis of the corresponding ideal. The proofs are based on easy computer calculations with standard functions of Maple, the explicit presentation of C_{32} in terms of generators and relations, and methods of representation theory of the general linear group.

MSC: 16R30; 16S15

Keywords: Generic matrices; Matrix invariants and concomitants; Trace algebras; Defining relations; Gröbner basis

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0. Introduction

Let K be any field of characteristic 0 and let $X_i = (x_{pq}^{(i)})$, $p, q = 1, \dots, n$, $i = 1, \dots, d$, be d generic $n \times n$ matrices. We consider the following two algebras: the pure (or commutative) trace algebra C_{nd} generated by all traces of products $\mathrm{tr}(X_{i_1} \cdots X_{i_k})$, and the mixed (or noncommutative) trace algebra T_{nd} generated by $X_1, \dots, X_d$ and C_{nd}, where we treat the elements of C_{nd} as scalar matrices. The algebra C_{nd} coincides with the algebra of invariants of the general linear group $GL_n = GL_n(K)$ acting by simultaneous conjugation on d matrices of size $n \times n$. The algebra T_{nd} is known as the algebra of matrix concomitants and also is the algebra of invariant functions under a suitable action of GL_n. General results of invariant theory of classical groups imply that the algebra C_{nd} is finitely generated and T_{nd} is a finitely generated C_{nd}-module. More precise results, see Van den Bergh [21], give that C_{nd} is a finitely generated free module of a polynomial subalgebra S. A similar result holds for T_{nd}. Theory of PI-algebras provides upper bounds for the generating sets of the algebra C_{nd} and of the C_{nd}-module T_{nd}. The Nagata–Higman theorem states that the polynomial identity $x^n = 0$ implies the identity $x_1 \cdots x_N = 0$ for some $N = N(n)$. If N is minimal with this property, then C_{nd} is generated by traces of products $\mathrm{tr}(X_{i_1} \cdots X_{i_k})$ of degree $k \leqslant N$ and T_{nd} is generated as a C_{nd}-module by products $X_{j_1} \cdots X_{j_l}$ of degree $l \leqslant N - 1$. These estimates are sharp if d is sufficiently large. A description of the defining relations of C_{nd} is given by the Razmyslov–Procesi theory [16,17] in the language of ideals of the group algebras of symmetric groups. For a background on the algebras of matrix invariants and concomitants see, e.g. [7,10].

Explicit minimal sets of generators of C_{nd} and T_{nd} and the defining relations between them are found in few cases only. It is well known that, in the Nagata–Higman theorem, $N(2) = 3$ and $N(3) = 6$, which gives bounds for the degrees of the generators of the algebras C_{2d} and C_{3d} and their modules T_{2d} and T_{3d}, respectively.

By a theorem of Sibirskii [18], C_{2d} is generated by $\mathrm{tr}(X_i)$, $1 \leqslant i \leqslant d$, $\mathrm{tr}(X_i X_j)$, $1 \leqslant i \leqslant j \leqslant d$, $\mathrm{tr}(X_i X_j X_k)$, $1 \leqslant i < j < k \leqslant d$. There are no relations between the five generators of the algebra C_{22}, i.e. $C_{22} \cong K[z_1, \dots, z_5]$. For $d = 3$, Sibirskii [18] found one relation and Formanek [9] proved that all relations follow from it. The center of GL_2 acts trivially by conjugation, and one has also a natural action of $PSL_2(K)$. Since $PSL_2(\mathbb{C})$ is isomorphic to $SO_3(\mathbb{C})$, one may apply invariant theory of orthogonal groups, see Procesi [16] and Le Bruyn [12]. Drensky [6] translated the generators and defining relations of the invariants of $SO_3(K)$ and obtained the defining relations of C_{2d} for all d. As a by-product of a result of Drensky and Koshlukov [8] on polynomial identities of Jordan algebras, one can easily obtain the defining relations of T_{2d}, see the comments in [6].

Teranishi [19] found the following system of generators of C_{32}:

$$\mathrm{tr}(X), \mathrm{tr}(Y), \mathrm{tr}(X^2), \mathrm{tr}(XY), \mathrm{tr}(Y^2),$$
$$\mathrm{tr}(X^3), \mathrm{tr}(X^2Y), \mathrm{tr}(XY^2), \mathrm{tr}(Y^3), \mathrm{tr}(X^2Y^2), \mathrm{tr}(X^2Y^2XY),$$

where X, Y are generic 3×3 matrices. He showed that the first ten of these generators form a homogeneous system of parameters of C_{32} and C_{32} is a free module with generators 1 and $\mathrm{tr}(X^2Y^2XY)$ over the polynomial algebra on these ten elements. Hence $\mathrm{tr}(X^2Y^2XY)$ satisfies a quadratic equation with coefficients depending on the other ten generators. The explicit (but very complicated) form of the equation was found by Nakamoto [15], over $\mathbb{Z}$, with respect to a slightly different system of generators. Abeasis and Pittaluga [1] found a system of generators of C_{3d}, for any $d \geqslant 2$, in terms of representation theory of the symmetric and general linear groups, in the spirit of its use in theory of PI-algebras. Teranishi [19] found also a set of generators and a homogeneous system of parameters of C_{42}. Recently, Aslaksen, Drensky and Sadikova [3] have found another natural set of eleven generators of the algebra C_{32} and have given the defining relation with respect to this set. Their relation is much simpler than that in [15].

The purpose of the present paper is to find a polynomial subalgebra S of C_{32} such that both C_{32} and T_{32} are finitely generated free S-modules and systems of free generators of them. We also give a system of generators and defining relations of T_{32} and find a Gröbner basis of the corresponding ideal with respect to a suitable ordering. The proofs are based on representation theory of GL_2 and use essentially the results of [3] combined with computer calculations with Maple. Although some of the main results (especially the Gröbner basis) are quite technical, we believe that they may serve as an "experimental material" giving some idea for the general picture, and the developed methods can be successfully used for further investigation of generic trace algebras.

1. Preliminaries

In what follows, we fix $n = 3$ and $d = 2$ and denote by X and Y the two generic 3×3 matrices. It is a standard trick to replace the generic matrices in T_{nd} with generic traceless matrices. We express X and Y in the form

$$X = \frac{1}{3}\operatorname{tr}(X)e + x \quad \text{and} \quad Y = \frac{1}{3}\operatorname{tr}(Y)e + y,$$

where e is the identity 3×3 matrix and x, y are generic traceless matrices. Then

$$C_{32} \cong K\big[\operatorname{tr}(X), \operatorname{tr}(Y)\big] \otimes_K C_0, \qquad T_{32} \cong K\big[\operatorname{tr}(X), \operatorname{tr}(Y)\big] \otimes_K T_0, \tag{1}$$

where the algebra C_0 is generated by the traces of products $\operatorname{tr}(z_1 \cdots z_k)$, $z_i = x, y$, $k \leqslant 6$. Moreover, T_0 is a C_0-module generated by the products $z_1 \cdots z_l$, $l \leqslant 5$. By well known arguments, as for "ordinary" generic matrices, without loss of generality we may assume that x is a diagonal matrix. Changing the variables x_{ii} and y_{ii}, we may assume that

$$x = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & -(x_1 + x_2) \end{pmatrix}, \qquad y = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & -(y_{11} + y_{22}) \end{pmatrix}. \tag{2}$$

Till the end of the paper we fix the notation x, y for these two generic traceless matrices. Now we state in detail the result of [3] in the form we need it in our paper. The generators of C_0 in [3] are $\operatorname{tr}(x^2), \operatorname{tr}(xy), \operatorname{tr}(y^2)$, $\operatorname{tr}(x^3), \operatorname{tr}(x^2y), \operatorname{tr}(xy^2), \operatorname{tr}(y^3)$, and the other two generators $\operatorname{tr}(x^2y^2), \operatorname{tr}(x^2y^2xy)$ of the system of Teranishi [19] are replaced by the elements

$$v = \operatorname{tr}\big(x^2y^2\big) - \operatorname{tr}(xyxy), \tag{3}$$

$$w = \operatorname{tr}\big(x^2y^2xy\big) - \operatorname{tr}\big(y^2x^2yx\big). \tag{4}$$

Consider the following elements of C_0:

$$u=\begin{vmatrix}\operatorname{tr}(x^2) & \operatorname{tr}(xy)\\ \operatorname{tr}(xy) & \operatorname{tr}(y^2)\end{vmatrix},$$

$$w_1=u^3,\quad w_2=u^2v,\quad w_4=uv^2,\quad w_7=v^3, \tag{5}$$

$$w_5=v\begin{vmatrix}\operatorname{tr}(x^2) & \operatorname{tr}(xy) & \operatorname{tr}(y^2)\\ \operatorname{tr}(x^3) & \operatorname{tr}(x^2y) & \operatorname{tr}(xy^2)\\ \operatorname{tr}(x^2y) & \operatorname{tr}(xy^2) & \operatorname{tr}(y^3)\end{vmatrix}, \tag{6}$$

$$w_6=\begin{vmatrix}\operatorname{tr}(x^3) & \operatorname{tr}(xy^2)\\ \operatorname{tr}(x^2y) & \operatorname{tr}(y^3)\end{vmatrix}^2-4\begin{vmatrix}\operatorname{tr}(y^3) & \operatorname{tr}(xy^2)\\ \operatorname{tr}(xy^2) & \operatorname{tr}(x^2y)\end{vmatrix}\begin{vmatrix}\operatorname{tr}(x^3) & \operatorname{tr}(x^2y)\\ \operatorname{tr}(x^2y) & \operatorname{tr}(xy^2)\end{vmatrix}, \tag{7}$$

$$w_3'=u\begin{vmatrix}\operatorname{tr}(x^2) & \operatorname{tr}(xy) & \operatorname{tr}(y^2)\\ \operatorname{tr}(x^3) & \operatorname{tr}(x^2y) & \operatorname{tr}(xy^2)\\ \operatorname{tr}(x^2y) & \operatorname{tr}(xy^2) & \operatorname{tr}(y^3)\end{vmatrix}, \tag{8}$$

$$\begin{aligned}w_3''&=5\big[\operatorname{tr}^3(y^2)\operatorname{tr}^2(x^3)+\operatorname{tr}^3(x^2)\operatorname{tr}^2(y^3)\big]\\&\quad-30\big[\operatorname{tr}^2(y^2)\operatorname{tr}(xy)\operatorname{tr}(x^2y)\operatorname{tr}(x^3)+\operatorname{tr}^2(x^2)\operatorname{tr}(xy)\operatorname{tr}(y^3)\operatorname{tr}(xy^2)\big]\\&\quad+3\big\{\big[4\operatorname{tr}(y^2)\operatorname{tr}^2(xy)+\operatorname{tr}^2(y^2)\operatorname{tr}(x^2)\big]\big[3\operatorname{tr}^2(x^2y)+2\operatorname{tr}(xy^2)\operatorname{tr}(x^3)\big]\\&\quad+\big[4\operatorname{tr}^2(xy)\operatorname{tr}(x^2)+\operatorname{tr}^2(x^2)\operatorname{tr}(y^2)\big]\big[3\operatorname{tr}^2(xy^2)+2\operatorname{tr}(x^2y)\operatorname{tr}(y^3)\big]\big\}\\&\quad-2\big[2\operatorname{tr}^3(xy)+3\operatorname{tr}(x^2)\operatorname{tr}(xy)\operatorname{tr}(y^2)\big]\big[9\operatorname{tr}(xy^2)\operatorname{tr}(x^2y)+\operatorname{tr}(x^3)\operatorname{tr}(y^3)\big].\end{aligned} \tag{9}$$

The element w_3'' can be expressed in another natural way. Recall that the K-linear operator δ of an algebra R is a derivation if $\delta(f_1f_2)=\delta(f_1)f_2+f_1\delta(f_2)$ for all $f_1,f_2\in R$. Let δ be the derivation of C_0 which commutes with the trace and is defined by $\delta(x)=0$, $\delta(y)=x$. Then

$$w_3''=\frac{1}{144}\sum_{i=0}^{6}(-1)^i\delta^i\big(\operatorname{tr}^3(y^2)\big)\delta^{6-i}\big(\operatorname{tr}^2(y^3)\big).$$

The main result of Aslaksen, Drensky and Sadikova [3] is that the algebra C_0 is generated by

$$\operatorname{tr}(x^2),\operatorname{tr}(xy),\operatorname{tr}(y^2),\operatorname{tr}(x^3),\operatorname{tr}(x^2y),\operatorname{tr}(xy^2),\operatorname{tr}(y^3),v,w$$

subject to the defining relation $f=0$, where

$$f=w^2-\left(\frac{1}{27}w_1-\frac{2}{9}w_2+\frac{4}{15}w_3'+\frac{1}{90}w_3''+\frac{1}{3}w_4-\frac{2}{3}w_5-\frac{1}{3}w_6-\frac{4}{27}w_7\right), \tag{10}$$

and the elements $v,w,w_1,w_2,w_3',w_3'',w_4,w_5,w_6,w_7$ are given in (3), (4), (5), (6), (7), (8) and (9). Let

$$S_0=K\big[\operatorname{tr}(x^2),\operatorname{tr}(xy),\operatorname{tr}(y^2),\operatorname{tr}(x^3),\operatorname{tr}(x^2y),\operatorname{tr}(xy^2),\operatorname{tr}(y^3),v\big]. \tag{11}$$

Then C_0 is a free S_0-module with basis $\{1,w\}$ and

$$C_{32}\cong K\big[\operatorname{tr}(X),\operatorname{tr}(Y)\big]\otimes_K S_0[w]/(f).$$

Recall that the algebras C_{nd} and T_{nd} have a natural multigrading which takes into account the degrees of the products $X_{j_1}\cdots X_{j_l}$ and of the traces $\mathrm{tr}(X_{i_1}\cdots X_{i_k})$ with respect to each of the generic matrices $X_1,\dots,X_d$. The Hilbert series of C_{nd} and T_{nd} are defined as the formal power series

$$H(C_{nd},t_1,\dots,t_d)=\sum_{k_i\geqslant 0}\dim\bigl(C_{nd}^{(k_1,\dots,k_d)}\bigr)t_1^{k_1}\cdots t_d^{k_d},$$
$$H(T_{nd},t_1,\dots,t_d)=\sum_{k_i\geqslant 0}\dim\bigl(T_{nd}^{(k_1,\dots,k_d)}\bigr)t_1^{k_1}\cdots t_d^{k_d},$$

with coefficients equal to the dimensions of the homogeneous components $C_{nd}^{(k_1,\dots,k_d)}$ and $T_{nd}^{(k_1,\dots,k_d)}$ of degree $(k_1,\dots,k_d)$, respectively. The Hilbert series carry a lot of information for the algebras. In the sequel we shall need the Hilbert series of C_{32} and T_{32} calculated, respectively, by Teranishi [19] and Berele and Stembridge [4]:

$$H(C_{32},t_1,t_2)=\frac{1+t_1^3t_2^3}{(1-t_1)(1-t_2)q_2(t_1,t_2)q_3(t_1,t_2)(1-t_1^2t_2^2)},$$
$$H(T_{32},t_1,t_2)=\frac{1}{(1-t_1)^2(1-t_2)^2(1-t_1^2)(1-t_2^2)(1-t_1t_2)^2(1-t_1^2t_2)(1-t_1t_2^2)},$$

where the commuting variables t_1 and t_2 count, respectively, the degrees of X and Y and

$$\begin{aligned}q_2(t_1,t_2)&=\bigl(1-t_1^2\bigr)(1-t_1t_2)\bigl(1-t_2^2\bigr),\\ q_3(t_1,t_2)&=\bigl(1-t_1^3\bigr)\bigl(1-t_1^2t_2\bigr)\bigl(1-t_1t_2^2\bigr)\bigl(1-t_2^3\bigr).\end{aligned}\tag{12}$$

Since the Hilbert series of the tensor product is equal to the product of the Hilbert series of the factors, and

$$H\bigl(K\bigl[\mathrm{tr}(X),\mathrm{tr}(Y)\bigr],t_1,t_2\bigr)=\frac{1}{(1-t_1)(1-t_2)},$$

(1) implies that

$$H(C_{32},t_1,t_2)=\frac{H(C_0,t_1,t_2)}{(1-t_1)(1-t_2)},\qquad H(T_{32},t_1,t_2)=\frac{H(T_0,t_1,t_2)}{(1-t_1)(1-t_2)}.$$

In this way,

$$H(C_0,t_1,t_2)=\frac{1+t_1^3t_2^3}{q_2(t_1,t_2)q_3(t_1,t_2)(1-t_1^2t_2^2)}.$$

We rewrite $H(T_0,t_1,t_2)$ in the form

$$H(T_0,t_1,t_2)=\frac{p(t_1,t_2)}{q_2(t_1,t_2)q_3(t_1,t_2)(1-t_1^2t_2^2)},\tag{13}$$

where

$$p(t_1,t_2)=\left(1+t_1+t_1^2\right)\left(1+t_2+t_2^2\right)(1+t_1t_2).$$

Now we summarize the necessary background on representation theory of GL_2. We refer e.g. to [13] for the general facts and to [5] for the applications in the spirit of the problems considered here. The irreducible polynomial representations of GL_2 are indexed by partitions $\lambda=(\lambda_1,\lambda_2)$, $\lambda_1\geqslant\lambda_2\geqslant 0$. We denote by $W(\lambda)$ the corresponding irreducible GL_2-module. The group GL_2 acts in the natural way on the two-dimensional vector space $K\cdot x+K\cdot y$ and this action is extended diagonally on the free associative algebra $K\langle x,y\rangle$. As a GL_2-module $K\langle x,y\rangle$ is completely reducible and

$$K\langle x,y\rangle=\sum_{\lambda}d(\lambda)W(\lambda),$$

where the multiplicity $d(\lambda)$ is equal to the degree of the corresponding S_k-module, $k=\lambda_1+\lambda_2$, and can be calculated using e.g. the hook formula. In particular,

$$d(\lambda_1,0)=1,\lambda_1\geqslant 0,\qquad d(\lambda_1,1)=\lambda_1-1,\lambda_1\geqslant 1,$$
$$d(2,2)=2,\qquad d(3,2)=5.$$

The module $W(\lambda)$ is generated by a unique, up to a multiplicative constant, homogeneous element w_λ of degree λ_1 and λ_2 with respect to x and y, respectively, called the highest weight vector of $W(\lambda)$. It is characterized by the following property, see Koshlukov [11] for the version which we need.

Lemma 1.1. *Let δ be the derivation of $K\langle x,y\rangle$ defined by $\delta(x)=0$, $\delta(y)=x$. If $w(x,y)\in K\langle x,y\rangle$ is homogeneous of degree (λ_1,λ_2), then $w(x,y)$ is a highest weight vector for some $W(\lambda_1,\lambda_2)$ if and only if $\delta(w(x,y))=0$.*

If W_i, $i=1,\dots,m$, are m isomorphic copies of $W(\lambda)$ and $w_i\in W_i$ are highest weight vectors, then the highest weight vector of any submodule $W(\lambda)$ of the direct sum $W_1\oplus\cdots\oplus W_m$ has the form $\xi_1w_1+\cdots+\xi_mw_m$ for some $\xi_i\in K$. Any m linearly independent highest weight vectors can serve as a set of generators of the GL_2-module $W_1\oplus\cdots\oplus W_m$. When $m=d(\lambda)$ and $W_1\oplus\cdots\oplus W_m\subset K\langle x,y\rangle$, then it is convenient to choose $w_1,\dots,w_m$ in the following way. For the standard λ-tableau

$$T_\sigma=\begin{array}{|c|c|c|c|c|c|}\hline \sigma(1) & \cdots & \sigma(2\lambda_2-1) & \sigma(2\lambda_2+1) & \cdots & \sigma(k)\\ \hline \sigma(2) & \cdots & \sigma(2\lambda_2) & \multicolumn{3}{c}{} \\ \cline{1-3}\end{array}$$

corresponding to $\sigma\in S_k$, $k=\lambda_1+\lambda_2$, we associate a highest weight vector $w(T_\sigma)$ in $K\langle x,y\rangle$. When $\sigma=\varepsilon$ is the identity of S_k we fix

$$\begin{aligned}w(T_\varepsilon)&=(xy-yx)^{\lambda_2}x^{\lambda_1-\lambda_2}\\&=\sum_{\rho_1,\dots,\rho_s\in S_2}\operatorname{sign}(\rho_1\cdots\rho_s)z_{\rho_1(1)}z_{\rho_1(2)}\cdots z_{\rho_s(1)}z_{\rho_s(2)}x^{\lambda_1-\lambda_2},\end{aligned}$$

where $z_1 = x$, $z_2 = y$, $s = \lambda_2$. For σ arbitrary, we define $w(T_\sigma)$ in a similar way, but the skew-symmetries are in positions $(\sigma(1), \sigma(2)), \ldots, (\sigma(2s-1), \sigma(2s))$ instead of the positions $(1,2), \ldots, (2s-1, 2s)$ (and the positions with fixed x are $\sigma(2s+1), \ldots, \sigma(k)$ instead of $2s+1, \ldots, k$). Recall also the Littlewood–Richardson rule, which in the case of GL_2 states

$$W(a+b,b) \otimes_K W(c+d,d) \cong \sum_{s=0}^{c} W_2(a+b+d+s, b+d+c-s), \quad a \geqslant c. \tag{14}$$

Finally, if W is a GL_2-submodule or a factor module of $K\langle x, y\rangle$, then W inherits the grading of $K\langle x, y\rangle$ and its Hilbert series plays the role of the GL_2-character of W: If

$$W \cong \sum_{\lambda} m(\lambda) W(\lambda),$$

then

$$H(W, t_1, t_2) = \sum_{\lambda} m(\lambda) S_\lambda(t_1, t_2),$$

where $S_\lambda = S_\lambda(t_1, t_2)$ is the Schur function associated with λ, and the multiplicities $m(\lambda)$ are determined by $H(W, t_1, t_2)$. In the case of two variables $S_{(\lambda_1,\lambda_2)}$ has the simple form

$$S_{(\lambda_1,\lambda_2)} = (t_1 t_2)^{\lambda_2}\bigl(t_1^{\lambda_1-\lambda_2} + t_1^{\lambda_1-\lambda_2-1} t_2 + \cdots + t_1 t_2^{\lambda_1-\lambda_2-1} + t_2^{\lambda_1-\lambda_2}\bigr).$$

The action of GL_2 on $K\langle x, y\rangle$ is inherited by the algebras C_{32}, T_{32}, C_0, and T_0. For example, the elements v and w in (3) and (4) generate one-dimensional GL_2-modules, isomorphic, respectively, to $W(2,2)$ and $W(3,3)$. As another example, the polynomial $p(t_1, t_2)$ from (13) has the form

$$\begin{aligned} p(t_1, t_2) = 1 &+ S_{(1,0)} + (S_{(2,0)} + S_{(1,1)}) \\ &+ 2S_{(2,1)} + (S_{(3,1)} + S_{(2,2)}) + S_{(3,2)} + S_{(3,3)}. \end{aligned} \tag{15}$$

Let $h_k = h_k(t_1, t_2)$ be the homogeneous component of degree k of the Hilbert series $H(C_0, t_1, t_2)$, i.e.

$$H(C_0, t_1, t_2) = h_0 + h_1 + h_2 + \cdots.$$

Direct calculations using the formula

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots$$

show that

$$h_0 = 1 = S_{(0,0)}, \quad h_1 = 0, \quad h_2 = t_1^2 + t_1t_2 + t_2^2 = S_{(2,0)},$$
$$h_3 = t_1^3 + t_1^2t_2 + t_1t_2^2 + t_2^3 = S_{(3,0)},$$
$$h_4 = t_1^4 + t_1^3t_2 + 3t_1^2t_2^2 + t_1t_2^3 + t_2^4 = S_{(4,0)} + 2S_{(2,2)}, \tag{16}$$
$$h_5 = S_{(5,0)} + S_{(4,1)} + S_{(3,2)}, \quad h_6 = 2S_{(6,0)} + 3S_{(4,2)} + S_{(3,3)},$$
$$h_7 = S_{(7,0)} + S_{(6,1)} + 3S_{(5,2)} + S_{(4,3)}.$$

2. Generators of the C_{32}-module T_{32}

In this section we modify for our purposes the idea of Abeasis and Pittaluga [1] used in their description of the generators of C_{nd}. We consider the C_0-module T_0. It has a system of generators of degree $\leqslant 5$. Without loss of generality we may assume that this system is in the K-subalgebra R_0 of T_0 generated by x, y. Let U_k be the C_0-submodule of T_0 generated by all products $z_1 \cdots z_l$ of degree $l \leqslant k$, $z_j = x, y$. Clearly, U_k is also a GL_2-submodule of T_0. Let $T_0^{(k+1)}$ be the homogeneous component of degree $k+1$ of T_0. Then the intersection $U_k \cap T_0^{(k+1)}$ is a GL_2-module and has a complement G_{k+1} in $T_0^{(k+1)}$, which is the GL_2-module of the "new" generators of degree $k+1$. We may assume that G_{k+1} is a GL_2-submodule of R_0. Then the GL_2-module of the generators of the C_0-module T_0 is

$$G = G_0 \oplus G_1 \oplus \cdots \oplus G_5. \tag{17}$$

We use the notation $[x, y] = xy - yx$ and $x \circ y = xy + yx$.

Lemma 2.1. *For $\lambda = (3, 1)$, the following polynomials in T_0 are highest weight vectors*:

$$u_1 = [x, y]x^2, \quad u_2 = \bigl[x^2, y\bigr]x, \quad u_3 = \bigl[x^3, y\bigr],$$
$$u_4 = \operatorname{tr}\bigl(x^2\bigr)(x \circ y) - 2\operatorname{tr}(xy)x^2, \quad u_5 = \operatorname{tr}\bigl(x^2\bigr)[x, y], \quad u_6 = \operatorname{tr}\bigl(x^3\bigr)y - \operatorname{tr}\bigl(x^2y\bigr)x.$$

The elements $u_1, \ldots, u_6$ satisfy the relations

$$2(u_1 + u_2) + u_4 - u_5 + 2u_6 = 0, \qquad 2u_3 - u_5 = 0. \tag{18}$$

Proof. The elements u_1, u_2, u_3 are the highest weight vectors corresponding, respectively, to the standard tableaux

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$$

The other elements u_4, u_5, u_6 are homogeneous of degree $(3, 1)$. Using Lemma 1.1, one checks that they are also highest weight vectors. In order to verify the relations, we have evaluated the expressions in (18) using Maple and assuming that x and y have the form (2). □

Lemma 2.2. *For* $\lambda = (2,2)$*, the following polynomials in* T_0 *are highest weight vectors*:

$$u_1 = [x,y]^2, \qquad u_2 = x\big[x,y^2\big] + y\big[y,x^2\big],$$
$$u_3 = \operatorname{tr}\big(x^2\big)y^2 - \operatorname{tr}(xy)(x \circ y) + \operatorname{tr}\big(y^2\big)x^2,$$
$$u_4 = \big(\operatorname{tr}\big(x^2\big)\operatorname{tr}\big(y^2\big) - \operatorname{tr}^2(xy)\big)e, \qquad u_5 = \big(\operatorname{tr}\big(x^2y^2\big) - \operatorname{tr}(xyxy)\big)e.$$

The elements $u_1, \ldots, u_5$ *satisfy the relation*

$$3w_1 - 3w_2 + 3w_3 - 2w_4 + 2w_5 = 0. \tag{19}$$

Proof. The considerations are similar to those in the proof of Lemma 2.2. The elements u_1, u_2 are the highest weight vectors corresponding, respectively, to the standard tableaux

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

and u_3, u_4, u_5 are homogeneous of degree $(2,2)$, satisfying the conditions of Lemma 1.1. Again, the relation (19) is obtained using Maple. □

Lemma 2.3. *For* $\lambda = (4,1)$*, the following polynomials in* T_0 *are highest weight vectors*:

$$u_1 = [x,y]x^3, \quad u_2 = \big[x^2,y\big]x^2, \quad u_3 = \big[x^3,y\big]x, \quad u_4 = \big[x^4,y\big],$$
$$u_5 = \operatorname{tr}\big(x^2\big)[x,y]x, \quad u_6 = \operatorname{tr}\big(x^2\big)\big[x^2,y\big],$$
$$u_7 = \operatorname{tr}\big(x^3\big)(x \circ y) - 2\operatorname{tr}\big(x^2y\big)x^2, \quad u_8 = \operatorname{tr}\big(x^3\big)[x,y],$$
$$u_9 = \operatorname{tr}\big(x^2\big)\big[\operatorname{tr}\big(x^2\big)y - \operatorname{tr}(xy)x\big], \quad u_{10} = \big[\operatorname{tr}\big(x^2\big)\operatorname{tr}\big(x^2y\big) - \operatorname{tr}(xy)\operatorname{tr}\big(x^3\big)\big]e.$$

The elements $u_1, \ldots, u_{10}$ *satisfy the relations*

$$\begin{gathered} 6u_1 - 3u_5 - 2u_8 = 0, \\ 6u_2 + u_5 - 2u_6 + 3u_7 - u_8 + u_9 + 2u_{10} = 0, \\ 2u_3 - u_5 = 0, \\ 6u_4 - 3u_6 - 2u_8 = 0. \end{gathered} \tag{20}$$

Proof. Again, u_1, u_2, u_3, u_4 are the highest weight vectors corresponding, respectively, to the standard tableaux

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 5 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 \\ \cline{1-1} \end{array}$$

and $u_5, \ldots, u_{10}$ are found with Lemma 1.1. The relations (20) are obtained using Maple. □

The proofs of the next two lemmas are similar.

Lemma 2.4. *For $\lambda=(3,2)$, the following polynomials in T_0 are highest weight vectors*:

$$u_1=[x,y]^2x,\quad u_2=\big(x\big[x,y^2\big]+y\big[y,x^2\big]\big)x,\quad u_3=[x,y]\big[x^2,y\big],$$
$$u_4=\big[x^2,yx\big]y+\big[y^2x,x\big]x,\quad u_5=\big[x^3,y\big]y+\big(y^2x^2-xyxy\big)x,$$
$$u_6=\operatorname{tr}\big(x^2\big)[x,y]y-\operatorname{tr}(xy)[x,y]x,\quad u_7=\operatorname{tr}\big(x^2\big)\big[x,y^2\big]-\operatorname{tr}(xy)\big[x^2,y\big],$$
$$u_8=\operatorname{tr}\big(x^3\big)y^2-\operatorname{tr}\big(x^2y\big)(x\circ y)+\operatorname{tr}\big(xy^2\big)x^2,$$
$$u_9=\big(\operatorname{tr}\big(x^2\big)\operatorname{tr}\big(y^2\big)-\operatorname{tr}^2(xy)\big)x,\quad u_{10}=\big(\operatorname{tr}\big(x^2y^2\big)-\operatorname{tr}(xyxy)\big)x,$$
$$u_{11}=\big[\operatorname{tr}\big(x^2\big)\operatorname{tr}\big(xy^2\big)-2\operatorname{tr}(xy)\operatorname{tr}\big(x^2y\big)+\operatorname{tr}\big(y^2\big)\operatorname{tr}\big(x^3\big)\big]e.$$

They satisfy the relations:

$$\begin{gathered}6u_1-6u_3+4u_6-2u_7-6u_8-2u_9+4u_{10}+u_{11}=0,\\ 6u_2-6u_3+2u_6+2u_7-6u_8-2u_9-u_{11}=0,\\ 6u_4-2u_6+4u_7+2u_{10}+u_{11}=0,\\ 6u_5-4u_6+2u_7-2u_{10}-u_{11}=0.\end{gathered}\tag{21}$$

In the above lemma the polynomials $u_1,\dots,u_5$ correspond, respectively, to the standard tableaux

$$\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}$$

Lemma 2.5. *For $\lambda=(4,3)$, the following polynomials in T_0 are highest weight vectors*:

$$u_0=\big[\operatorname{tr}\big(x^2y^2xy\big)-\operatorname{tr}\big(y^2x^2yx\big)\big]x,$$
$$\begin{aligned}u_1=\big[&\operatorname{tr}(xy)\operatorname{tr}\big(y^2\big)\operatorname{tr}\big(x^3\big)-2\operatorname{tr}^2(xy)\operatorname{tr}\big(x^2y\big)-\operatorname{tr}\big(x^2\big)\operatorname{tr}\big(y^2\big)\operatorname{tr}\big(x^2y\big)\\&+3\operatorname{tr}\big(x^2\big)\operatorname{tr}(xy)\operatorname{tr}\big(xy^2\big)-\operatorname{tr}^2\big(x^2\big)\operatorname{tr}\big(y^3\big)\big]e,\end{aligned}$$
$$u_2=\big(\operatorname{tr}\big(x^2\big)\operatorname{tr}\big(y^2\big)-\operatorname{tr}^2(xy)\big)\big[\operatorname{tr}\big(x^2\big)y-\operatorname{tr}(xy)x\big],$$
$$u_3=\big(\operatorname{tr}\big(x^2y^2\big)-\operatorname{tr}(xyxy)\big)\big[\operatorname{tr}\big(x^2\big)y-\operatorname{tr}(xy)x\big],$$
$$u_4=\operatorname{tr}\big(x^3\big)\big[\operatorname{tr}\big(y^3\big)x-2\operatorname{tr}\big(xy^2\big)y\big]-\operatorname{tr}\big(x^2y\big)\big[\operatorname{tr}\big(xy^2\big)x-2\operatorname{tr}\big(x^2y\big)y\big],$$
$$\begin{aligned}u_5=\ &\operatorname{tr}\big(x^2\big)\big[\operatorname{tr}\big(y^3\big)x^2-\operatorname{tr}\big(xy^2\big)(x\circ y)+\operatorname{tr}\big(x^2y\big)y^2\big]\\&-\operatorname{tr}(xy)\big[\operatorname{tr}\big(xy^2\big)x^2-\operatorname{tr}\big(x^2y\big)(x\circ y)+\operatorname{tr}\big(x^3\big)y^2\big],\end{aligned}$$
$$\begin{aligned}u_6=\ &2\big[\operatorname{tr}(xy)\operatorname{tr}\big(xy^2\big)-\operatorname{tr}\big(y^2\big)\operatorname{tr}\big(x^2y\big)\big]x^2\\&-\big[\operatorname{tr}\big(x^2\big)\operatorname{tr}\big(xy^2\big)-\operatorname{tr}\big(y^2\big)\operatorname{tr}\big(x^3\big)\big](x\circ y)\\&+2\big[\operatorname{tr}\big(x^2\big)\operatorname{tr}\big(x^2y\big)-\operatorname{tr}(xy)\operatorname{tr}\big(x^3\big)\big]y^2,\end{aligned}$$

$$u_7 = \left[\operatorname{tr}(x^2)\operatorname{tr}(xy^2) - 2\operatorname{tr}(xy)\operatorname{tr}(x^2y) + \operatorname{tr}(y^2)\operatorname{tr}(x^3)\right][x,y],$$
$$u_8 = \left[\operatorname{tr}(x^2)\operatorname{tr}(y^2) - \operatorname{tr}^2(xy)\right][x,y]x,$$
$$u_9 = \left[\operatorname{tr}(x^2y^2) - \operatorname{tr}(xyxy)\right][x,y]x,$$
$$u_{10} = \left[\operatorname{tr}(x^2)\operatorname{tr}(y^2) - \operatorname{tr}^2(xy)\right]\left[x^2,y\right],$$
$$u_{11} = \left[\operatorname{tr}(x^2y^2) - \operatorname{tr}(xyxy)\right]\left[x^2,y\right],$$
$$u_{12} = \operatorname{tr}(x^3)[x,y]y^2 - \operatorname{tr}(x^2y)[x,y](x\circ y) + \operatorname{tr}(xy^2)[x,y]x^2,$$
$$u_{13} = \operatorname{tr}(x^2)[x,y]^2y - \operatorname{tr}(xy)[x,y]^2x,$$

and satisfy the relation

$$\begin{aligned} &18u_0 - 4u_1 - 2u_2 + 10u_3 + 18u_4 - 6u_5 \\ &+ 9u_6 + 15u_7 - 4u_8 - 4u_{10} + 12u_{11} - 36u_{12} + 12u_{13} = 0. \end{aligned} \tag{22}$$

Proposition 2.6. (i) *The GL_2-module $G = G_0 \oplus \cdots \oplus G_5$ generates T_0 as a C_0-module, where $G_0, G_1, G_2, G_3, G_4, G_5$ are generated by the sets of elements $\{u_{(0,0)}\}$, $\{u_{(1,0)}\}$, $\{u_{(2,0)}, u_{(1,1)}\}$, $\{u'_{(2,1)}, u''_{(2,1)}\}$, $\{u_{(3,1)}, u_{(2,2)}\}$, and $\{u_{(3,2)}\}$, respectively, and*

$$u_{(0,0)} = 1, \quad u_{(1,0)} = x, \quad u_{(2,0)} = x^2, \quad u_{(1,1)} = [x,y],$$
$$u'_{(2,1)} = [x,y]x, \quad u''_{(2,1)} = \left[x^2,y\right],$$
$$u_{(3,1)} = [x,y]x^2, \quad u_{(2,2)} = [x,y]^2, \quad u_{(3,2)} = [x,y]^2x.$$

(ii) *Let $G_6 = Kw$, where $w \in C_0$ is defined in* (4). *Then $G \oplus G_6$ generates T_0 as an S_0-module, where S_0 is the subalgebra of C_0 defined in* (11).

Proof. (i) Let $R_0^{(k)}$ be the homogeneous component of degree k of the K-subalgebra R_0 of T_0 generated by x, y. It is sufficient to show that every irreducible GL_2-submodule $W(\lambda)$ of $R_0^{(k)}$, $k = \lambda_1 + \lambda_2 \leqslant 5$, belongs to C_0G. For $k \leqslant 2$ this is obvious because $R_0^{(k)} = G_k$. Similarly, any $W(2,1) \subset R_0^{(3)}$ is contained in the GL_2-submodule generated by $u'_{(2,1)}, u''_{(2,1)}$. The submodule $W(3,0)$ of $R_0^{(3)}$ is generated by x^3. We use the equation

$$x^3 - \frac{1}{2}\operatorname{tr}(x^2)x - \frac{1}{3}\operatorname{tr}(x^3)e = 0, \tag{23}$$

which follows from the Cayley–Hamilton theorem and derive that $W(3,0) \subset C_0G_1 + C_0G_0$. This implies that x^4 and x^5, and hence $W(4,0)$ and $W(5,0)$, also belong to C_0G.

Let $\lambda = (3,1)$. In the notation of Lemma 2.1, the element $u_1 = [x,y]x^2$ coincides with $u_{(3,1)}$, hence belongs to C_0G. The elements u_4, u_5, u_6 also belong to C_0G. Therefore the relations (18) give that u_2 and u_3 are in C_0G. Since every GL_2-submodule $W(3,1)$ of $R_0^{(4)}$ is contained in the GL_2-module generated by the highest weight vectors u_1, u_2, u_3, we obtain that every $W(3,1) \subset R_0^{(4)}$ is in C_0G.

Let $\lambda = (2,2)$. The element $u_1 = [x,y]^2$ in Lemma 2.2 is the same as $u_{(2,2)}$ and hence belongs to C_0G. The elements u_3, u_4, u_5 also belong to C_0G. The relation (19) gives that $u_2 \in C_0G$ and this completes the case $\lambda = (2,2)$ because every $W(2,2) \subset R_0^{(4)}$ is a submodule of the GL_2-module generated by u_1 and u_2.

The cases $\lambda = (4,1)$ and $\lambda = (3,2)$ are similar. In the former case, the relations (20) in Lemma 2.3 give that all highest weight vectors u_1, u_2, u_3, u_4 are linear combinations of $u_5, \ldots, u_{10}$ and are in C_0G. In the latter case, the element $u_1 = [x,y]^2x$ from Lemma 2.4 is equal to $u_{(3,2)}$. Hence the relations (21) give that u_2, u_3, u_4, u_5 are linear combinations of $u_1 = u_{(3,2)}$ and $u_6, \ldots, u_{11}$. In this way, every $W(3,2) \subset R_0^{(5)}$ is in C_0G.

(ii) Since C_0 is a free S_0-module with basis $\{1, w\}$, in virtue of (i), it is sufficient to show that

$$S_0wG \subset \sum_{k=0}^{5} S_0wR_0^{(k)} \subset S_0G + S_0w.$$

For $R_0^{(0)} = G_0 = K$ we obtain immediately $S_0wR_0^{(0)} = S_0w \subset S_0G + S_0w$. The element w generates the one-dimensional GL_2-module Kw isomorphic to $W(3,3)$. The GL_2-module $R_0^{(1)} = G_1 \cong W(1,0)$ is generated by x and has a basis $\{x,y\}$. The Littlewood–Richardson rule gives that $W(3,3) \otimes_K W(1,0) \cong W(4,3)$. The module $W(4,3)$ is generated by wx and is spanned by $\{wx, wy\}$. The explicit form of the elements $u_1, \ldots, u_{13}$ in Lemma 2.5 shows that they belong to S_0G, and the relation (22) gives that $u_0 = wx$ is their linear combination, hence also belongs to S_0G. In this way, $R_0^{(1)}wG_1 = S_0wx + S_0wy \subset S_0G$. For any product $a = z_1 \cdots z_k$ of degree $k = 1, \ldots, 5$, where $z_j = x, y$, by (i), we know that $a = a' + a'' \in S_0G + S_0wG$, where $a' \in S_0G$ and $a'' \in S_0wG$. Since T_0 is a graded vector space, and S_0G, S_0wG are its graded subspaces, the inequality $\deg(a) \leqslant 5 < 6 = \deg(w)$ gives that $a'' = 0$, i.e. $a \in S_0G$. Also, any product $b = z_1 \cdots z_6$, $z_j = x, y$, belongs to $S_0G + S_0wG$. Since $\deg(b) = 6$, and the only elements of degree 6 in S_0wG are Kw, we obtain that

$$b \in S_0G + Kw \subset \sum_{l=0}^{5} S_0R_0^{(l)} + Kw.$$

From $wz_1 \in \sum_{l=0}^{5} S_0R_0^{(l)}$ and $R_0^{(6)} \subset \sum_{l=0}^{5} S_0R_0^{(l)} + S_0w$ we derive

$$wz_1z_2 \in \left(\sum_{l=0}^{5} S_0R_0^{(l)}\right)z_2 \subset \sum_{l=1}^{6} S_0R_0^{(l)} \subset \sum_{l=0}^{5} S_0R_0^{(l)} + S_0R_0^{(6)} \subset \sum_{l=0}^{5} S_0R_0^{(l)} + S_0w.$$

Continuing in this way, we obtain that

$$wz_1 \cdots z_k \in \sum_{l=0}^{5} S_0R_0^{(l)} + S_0w, \quad k \leqslant 5,$$

which completes the proof. $\square$

Now we state the main result of this section.

Theorem 2.7. *Let T_{32} be the mixed trace algebra generated by the generic 3×3 matrices X, Y, and let x, y be the generic traceless matrices*

$$x = X - \frac{1}{3}\operatorname{tr}(X)e, \qquad y = Y - \frac{1}{3}\operatorname{tr}(Y)e.$$

Let $G = G_0 \oplus G_1 \oplus \cdots \oplus G_5$, where G_k is the GL_2-module generated by the elements of degree k among

$$u_{(0,0)} = 1, \quad u_{(1,0)} = x, \quad u_{(2,0)} = x^2, \quad u_{(1,1)} = [x, y],$$
$$u'_{(2,1)} = [x, y]x, \quad u''_{(2,1)} = \left[x^2, y\right],$$
$$u_{(3,1)} = [x, y]x^2, \quad u_{(2,2)} = [x, y]^2, \quad u_{(3,2)} = [x, y]^2 x.$$

(i) *As a GL_2-module*

$$G \cong W(0,0) \oplus W(1,0) \oplus W(2,0) \oplus W(1,1)$$
$$\oplus 2W(2,1) \oplus W(3,1) \oplus W(2,2) \oplus W(3,2).$$

The vector space G generates T_{32} as a C_{32}-module.

(ii) *Let*

$$v = \operatorname{tr}\left(x^2y^2\right) - \operatorname{tr}(xyxy), \qquad w = \operatorname{tr}\left(x^2y^2xy\right) - \operatorname{tr}\left(y^2x^2yx\right),$$
$$S = K\left[\operatorname{tr}(X), \operatorname{tr}(Y), \operatorname{tr}\left(x^2\right), \operatorname{tr}(xy), \operatorname{tr}\left(y^2\right), \operatorname{tr}\left(x^3\right), \operatorname{tr}\left(x^2y\right), \operatorname{tr}\left(xy^2\right), \operatorname{tr}\left(y^3\right), v\right], \tag{24}$$

$G_6 = Kw$. Then S is isomorphic to the polynomial algebra in ten variables and T_{32} is a free S-module. Any basis of the vector space $G \oplus G_6$ serves as a set of free generators of T_{32}.

Proof. Combining Proposition 2.6(i) and the equality (1) we obtain that G generates the C_{32}-module T_{32}. Together with the equality (11) this gives that S is isomorphic to the polynomial algebra in ten variables and T_{32} is an S-module generated by $G \oplus G_6$. Hence, as a graded vector space, T_{32} is a homomorphic image of the tensor product $(G \oplus G_6) \otimes_K S$ and $G \oplus G_6$ is a homomorphic image of

$$W(0,0) \oplus W(1,0) \oplus W(2,0) \oplus W(1,1)$$
$$\oplus 2W(2,1) \oplus W(3,1) \oplus W(2,2) \oplus W(3,2).$$

Hence the Hilbert series of T_{32} and $G \otimes_K S$ coincide if and only if they are isomorphic as graded vector spaces. The Hilbert series of $W(0,0) \oplus W(1,0) \oplus W(2,0) \oplus W(1,1) \oplus 2W(2,1) \oplus W(3,1) \oplus W(2,2) \oplus W(3,2)$ is equal to

$$\begin{aligned}&1+S_{(1,0)}+(S_{(2,0)}+S_{(1,1)})\\&\quad+2S_{(2,1)}+(S_{(3,1)}+S_{(2,2)})+S_{(3,2)}+S_{(3,3)},\end{aligned}$$

and this is the expression of $p(t_1,t_2)$ from (15). The Hilbert series of S is

$$H(S,t_1,t_2)=\prod_{i=1}^{10}\frac{1}{1-t_1^{a_i}t_2^{b_i}},$$

where (a_i,b_i) are the degrees of the generators of S. Hence the denominator of $H(S,t_1,t_2)$ is

$$\begin{gathered}(1-t_1)(1-t_2)\big(1-t_1^2\big)(1-t_1t_2)\big(1-t_2^2\big)\\ \big(1-t_1^3\big)\big(1-t_1^2t_2\big)\big(1-t_1t_2^2\big)\big(1-t_2^3\big)\big(1-t_1^2t_2^2\big).\end{gathered}$$

This shows that both Hilbert series coincide and T_{32} is a free S-module generated by any basis of the vector space $G\oplus G_6$. □

In the end of the section we shall explain how to find the elements u_i in Lemmas 2.1, 2.2, 2.3, 2.4, and 2.5, and the relations between them. We shall illustrate this on Lemma 2.1. Assume that we already know that the generators (17) of degree $\leqslant 3$ of the C_0-module T_0 are

$$G_0=K\cong W(0,0),\quad G_1\cong W(1,0),\quad G_2=W(2,0)\oplus W(1,1),\quad G_3\cong 2W(2,1).$$

(Compare with (15)!) Clearly,

$$T_0^{(4)}=G_4\oplus\big(C_0^{(1)}G_3+C_0^{(2)}G_2+C_0^{(3)}G_1+C_0^{(4)}G_0\big)$$

(the sum in the parentheses may be not direct). Hence $T_0^{(4)}$ is a homomorphic image of the GL_2-module

$$K\langle x,y\rangle^{(4)}\oplus\big(C_0^{(1)}\otimes_K G_3\big)\oplus\big(C_0^{(2)}\otimes_K G_2\big)\oplus\big(C_0^{(3)}\otimes_K G_1\big)\oplus\big(C_0^{(4)}\otimes_K G_0\big).$$

Using (16) we derive

$$C_0^{(1)}=0,\quad C_0^{(2)}\cong W(2,0),\quad C_0^{(3)}\cong W(3,0),\quad C_0^{(4)}\cong W(4,0)\oplus W(2,2).$$

The Littlewood–Richardson rule (14) gives that

$$\begin{gathered}C_0^{(2)}\otimes_K G_2\cong W(4,0)\oplus 2W(3,1)\oplus W(2,2),\\ C_0^{(3)}\otimes_K G_1\cong W(4,0)\oplus W(3,1).\end{gathered}$$

In this way we prove the existence of the highest weight vectors $u_4,u_5\in C_0^{(2)}G_2$ and $u_6\in C_0^{(3)}G_1$. The component $C_0^{(4)}G_0$ does not contain a submodule $W(3,1)$. From the equality

$$K\langle x,y\rangle^{(4)}\cong W(4,0)\oplus 3W(3,1)\oplus 2W(2,2)$$

we obtain h_1, h_2, h_3. The next step is to find the relations (18). We consider the matrix equation

$$r_{(3,1)} = \xi_1 u_1 + \xi_2 u_2 + \xi_3 u_3 + \xi_4 u_4 + \xi_5 u_5 + \xi_6 u_6 = 0 \tag{25}$$

with unknowns $\xi_1, \dots, \xi_6 \in K$. The entries $(r_{(3,1)})_{ab}$ of the 3×3 matrix $r_{(3,1)}$ are polynomials in x_i, y_{ij}. Each monomial of $(r_{(3,1)})_{ab}$ is a linear combination of $\xi_1, \dots, \xi_6$ which gives an "ordinary" homogeneous linear equation between the ξs. We have established that all solutions of (25) are linear combinations of (18). In our exposition we have not used that we know that (18) give all the solutions and have derived this with other, "computer free" considerations. The situation is similar for the cases $\lambda = (2,2), (4,1), (3,2), (4,3)$, when all the relations between the highest weight vectors in T_0 follow, respectively, from (19), (20), (21), (22).

3. The Gröbner basis

In this section we give the Gröbner basis of the ideal generated by the defining relations of the algebra T_{32} with respect to a system of generators and an ordering chosen in a suitable way. Usually Gröbner bases are defined for ideals of polynomial algebras and free associative algebras with coefficients from a field, see e.g. [2] and [20]. In our case we prefer to state the result in terms of ideals of free algebras with coefficients from polynomial algebras. We use some formalism in the spirit of the one considered by Mikhalev and Zolotykh [14].

Let $U = \{u_1, \dots, u_p\}$ and $V = \{v_1, \dots, v_q\}$ be linearly ordered finite sets and let $[U]$ and $\langle V \rangle$ be the free abelian semigroup and the free noncommutative semigroup generated by U and V, respectively. The elements of $[U]$ are the "usual monomials" $u_1^{k_1} \cdots u_p^{k_p}$ and the elements of $\langle V \rangle$ are the words $v_{j_1} \cdots v_{j_s}$, with the "usual" multiplication of noncommutative elements, as in the free associative algebra. We consider the direct product $[U]\langle V \rangle$ with an arbitrary total ordering which extends the orderings of U and V, satisfies the descending chain condition, and is a monoid ordering. The latter means that if $af > bg$ for some $a, b \in [U]$ and $f, g \in \langle V \rangle$, then $acfh > bcgh$ and $achf > bchg$ for all $c \in [U]$ and $h \in \langle V \rangle$. We say that af divides bg, if there exist $c \in [U]$ and $h_1, h_2 \in \langle V \rangle$ such that $bg = cah_1fh_2$. We call the elements of $[U]\langle V \rangle$ (generalized) monomials. Every nonzero element of the free associative algebra

$$K[U]\langle V \rangle = K[u_1, \dots, u_p]\langle v_1, \dots, v_q \rangle$$

with polynomial coefficients from $K[U]$ is a finite sum of the form

$$z = \sum_{a_i \in [U]} \sum_{f_j \in \langle V \rangle} \alpha_{ij} a_i f_j, \quad \alpha_{ij} \in K, \quad a_1 f_1 > a_2 f_2 > \cdots.$$

We denote by z^0 the leading monomial $a_1 f_1$ of z.

Definition 3.1. Let I be a two-sided ideal of $K[U]\langle V \rangle$ and let I^0 be the set of leading monomials of I. A subset B of I is called a Gröbner basis of I (with respect to the fixed total ordering on $[U]\langle V \rangle$) if for any $z \in I$ there exists a $z_i \in B$ such that z^0 is divisible by z_i^0. Equivalently, the set B^0 generates the semigroup ideal I^0 of $[U]\langle V \rangle$.

The Gröbner basis B has the property that the subset of $[U]\langle V \rangle$ of all generalized monomials, which are not divisible by an element of B^0, forms a K-basis of the factor algebra $K[U]\langle V \rangle / I$.

Till the end of the paper we fix the set U to consist of the commuting variables

$$u_{10}, u_{01}, u_{20}, u_{11}, u_{02}, u_{30}, u_{21}, u_{12}, u_{03}, u_{22}, \tag{26}$$

and V to consist of the noncommuting variables x_1, y_1, w_{33}. Then the algebra T_{32} is a homomorphic image of $K[U]\langle V\rangle$, under the homomorphism π defined by

$$\begin{gathered}\pi : u_{10} \to \operatorname{tr}(X), \quad \pi : u_{01} \to \operatorname{tr}(Y),\\ \pi : u_{ij} \to \operatorname{tr}\bigl(x^i y^j\bigr), \quad i+j = 2,3, \quad \pi : u_{22} \to v,\\ \pi : x_1 \to x, \quad \pi : y_1 \to y, \quad \pi : w_{33} \to w,\end{gathered} \tag{27}$$

where the elements v and w are defined, respectively, in (3) and (4).

We define an arbitrary total ordering on $[U]$, with the only restrictions that it satisfies the descending chain condition and is compatible with the multiplication. We assume that $\deg(x_1) = \deg(y_1) = 1$ and $\deg(w_{33}) = 6$ and order the elements of $\langle V\rangle$ in the deg-lex way: If $f, g \in \langle V\rangle$ and $\deg(f) > \deg(g)$, then $f > g$ (first by degree). If $\deg(f) = \deg(g)$, then we order f and g lexicographically assuming that $w_{33} > x_1 > y_1$. Finally, we assume that $af > bg$, $a, b \in [U]$, $f, g \in \langle V\rangle$, if $f > g$ or, if $f = g$, then $a > b$. Our purpose is to find the Gröbner basis of the ideal $\operatorname{Ker}(\pi)$ with respect to this ordering. We need some more relations in T_{32} which have been found and verified using Maple.

Lemma 3.2. *The following relations hold in the algebra T_0:*

$$\begin{aligned}&6(xy)^2 - 6y^2x^2 + 3\operatorname{tr}\bigl(y^2\bigr)x^2 - 6\operatorname{tr}(xy)xy + 3\operatorname{tr}\bigl(x^2\bigr)y^2\\ &\quad - 2\bigl(\operatorname{tr}\bigl(x^2\bigr)\operatorname{tr}\bigl(y^2\bigr) - \operatorname{tr}^2(xy)\bigr)e + 2\bigl(\operatorname{tr}\bigl(x^2y^2\bigr) - \operatorname{tr}(xyxy)\bigr)e = 0, \qquad (28)\\ &36y^2xyx^2 - 6\operatorname{tr}\bigl(y^2\bigr)xyx^2 + 12\operatorname{tr}(xy)(yx)^2 - 12\operatorname{tr}(xy)y^2x^2 - 6\operatorname{tr}\bigl(x^2\bigr)y^2xy\\ &\quad + 12\operatorname{tr}\bigl(xy^2\bigr)xyx - 12\operatorname{tr}\bigl(xy^2\bigr)yx^2 + 12\operatorname{tr}\bigl(x^2y\bigr)yxy - 12\operatorname{tr}\bigl(x^2y\bigr)y^2x\\ &\quad + \bigl(-\operatorname{tr}\bigl(x^2\bigr)\operatorname{tr}\bigl(y^2\bigr) + 4\operatorname{tr}^2(xy) + 4\bigl(\operatorname{tr}\bigl(x^2y^2\bigr) - \operatorname{tr}(xyxy)\bigr)\bigr)xy\\ &\quad + 2\bigl(-\operatorname{tr}\bigl(x^2\bigr)\operatorname{tr}\bigl(y^2\bigr) - 2\operatorname{tr}^2(xy) + 4\bigl(\operatorname{tr}\bigl(x^2y^2\bigr) - \operatorname{tr}(xyxy)\bigr)\bigr)yx\\ &\quad + 2\bigl(\operatorname{tr}\bigl(x^2\bigr)\operatorname{tr}\bigl(y^3\bigr) - 6\operatorname{tr}(xy)\operatorname{tr}\bigl(xy^2\bigr) + 3\operatorname{tr}\bigl(y^2\bigr)\operatorname{tr}\bigl(x^2y\bigr)\bigr)x\\ &\quad + 2\bigl(\operatorname{tr}\bigl(y^2\bigr)\operatorname{tr}\bigl(x^3\bigr) - 6\operatorname{tr}(xy)\operatorname{tr}\bigl(x^2y\bigr) + 3\operatorname{tr}\bigl(x^2\bigr)\operatorname{tr}\bigl(xy^2\bigr)\bigr)y\\ &\quad + \bigl(2\operatorname{tr}\bigl(x^2\bigr)\operatorname{tr}(xy)\operatorname{tr}\bigl(y^2\bigr) - \operatorname{tr}^3(xy) + \operatorname{tr}\bigl(x^3\bigr)\operatorname{tr}\bigl(y^3\bigr) - 3\operatorname{tr}\bigl(x^2y\bigr)\operatorname{tr}\bigl(xy^2\bigr)\\ &\quad - \bigl(\operatorname{tr}\bigl(x^2y^2\bigr) - \operatorname{tr}(xyxy)\bigr)\operatorname{tr}(xy) - 3\bigl(\operatorname{tr}\bigl(x^2y^2xy\bigr) - \operatorname{tr}\bigl(y^2x^2yx\bigr)\bigr)\bigr)e = 0. \qquad (29)\end{aligned}$$

The partial linearizations of the Cayley–Hamilton theorem for 3×3 traceless matrices (23) give the following relations in T_0

$$\begin{gathered}x^2y + xyx + yx^2 - \operatorname{tr}(xy)x - \tfrac{1}{2}\operatorname{tr}\bigl(x^2\bigr)y - \operatorname{tr}\bigl(x^2y\bigr)e = 0,\\ xy^2 + yxy + y^2x - \tfrac{1}{2}\operatorname{tr}\bigl(y^2\bigr)x - \operatorname{tr}(xy)y - \operatorname{tr}\bigl(xy^2\bigr)e = 0,\\ y^3 - \tfrac{1}{2}\operatorname{tr}\bigl(y^2\bigr)y - \tfrac{1}{3}\operatorname{tr}\bigl(y^3\bigr)e = 0.\end{gathered} \tag{30}$$

We shall need the following easy combinatorial lemma.

Lemma 3.3. *The only words of degree* $\geqslant 3$ *in* $\langle x_1, y_1\rangle$ *which do not contain as a subword any of the words*

$$x_1^3, \quad x_1^2y_1, \quad x_1y_1^2, \quad y_1^3, \quad x_1y_1x_1y_1, \quad y_1^2x_1y_1x_1^2 \tag{31}$$

are

$$\begin{gathered} x_1y_1x_1, \quad y_1x_1^2, \quad y_1x_1y_1, \quad y_1^2x_1, \\ x_1y_1x_1^2, \quad y_1x_1y_1x_1, \quad y_1^2x_1^2, \quad y_1^2x_1y_1, \\ y_1x_1y_1x_1^2, \quad y_1^2x_1y_1x_1. \end{gathered} \tag{32}$$

Proof. The words from (31) are of degree $\leqslant 6$ and these from (32) are of degree $\leqslant 5$. Hence, it is sufficient to show that all words of degree 3, 4, 5 and 6 which do not contain (31) are those from (32). This can be done by direct verification. For example, we give the list of all words of degree 4, writing in parentheses the subwords from (31):

$$\begin{gathered} (x_1^3)x_1, \quad (x_1^3)y_1, \quad (x_1^2y_1)x_1, \quad x_1y_1x_1^2, \quad y_1(x_1^3), \\ (x_1^2y_1)y_1, \quad (x_1y_1x_1y_1), \quad (x_1y_1^2)x_1, \quad y_1(x_1^2y_1), \quad y_1x_1y_1x_1, \quad y_1^2x_1^2, \\ x_1(y_1^3), \quad y_1(x_1y_1^2), \quad y_1^2x_1y_1, \quad (y_1^3)x_1, \quad (y_1^3)y_1. \qquad \square \end{gathered}$$

The homomorphism π from (27) maps bijectively the polynomial algebra in ten variables $K[U]$ to S, where S is from (24) and U is the set of variables (26). We define an isomorphism $\rho: S \to K[U]$ as the inverse of the isomorphism $\pi|_{K[U]}: K[U] \to S$, i.e.

$$\begin{gathered} \rho: \mathrm{tr}(X) \to u_{10}, \quad \rho: \mathrm{tr}(Y) \to u_{01}, \\ \rho: \mathrm{tr}(x^i y^j) \to u_{ij}, \quad i+j = 2, 3, \quad \rho: v \to u_{22}. \end{gathered}$$

Any element of the subalgebra SR_0 of T_{32}, where R_0 is generated by x, y, has the form

$$f = \sum a_z z_1 \cdots z_l, \quad a_z \in S, \quad z_j = x, y.$$

It will be convenient, for a fixed presentation of f, to denote by $\rho(f)$ the element

$$\rho(f) = \sum \rho(a_z)\rho(z_1)\cdots\rho(z_l) \in K[U]\langle x_1, y_1, w_{33}\rangle,$$

where $\rho(x) = x_1$, $\rho(y) = y_1$, (and $\rho(e) = 1$). Pay attention, that $\rho(f)$ depends on the concrete form of f.

Now we give the elements of $K[U]\langle x_1, y_1, w_{33}\rangle$ which will be included in the Gröbner basis of $\mathrm{Ker}(\pi)$.

The first two elements are

$$f_1 = w_{33}x_1 - x_1w_{33}, \quad f_2 = w_{33}y_1 - y_1w_{33}. \tag{33}$$

Let $w_1, w_2, w_3', w_3'', w_4, w_5, w_6, w_7$ be the elements from (5), (6), (7), (8) and (9). We use the relation (10) and construct

$$f_3 = w_{33}^2 - \rho\left(\frac{1}{27}w_1 - \frac{2}{9}w_2 + \frac{4}{15}w_3' + \frac{1}{90}w_3'' + \frac{1}{3}w_4 - \frac{2}{3}w_5 - \frac{1}{3}w_6 - \frac{4}{27}w_7\right). \quad (34)$$

The element u_0 from Lemma 2.5 is equal to xw. We rewrite (22) as

$$f_{43}(x,y) = 18xw + \sum_{j=1}^{13}\alpha_j u_j,$$

$$\begin{aligned}\sum_{j=1}^{13}\alpha_j u_j = {} & -4u_1 - 2u_2 + 10u_3 + 18u_4 - 6u_5 \\ & + 9u_6 + 15u_7 - 4u_8 - 4u_{10} + 12u_{11} - 36u_{12} + 12u_{13}.\end{aligned}$$

Since $w(y,x) = -w(x,y)$, we derive that $f_{43}(y,x)$ has the form

$$f_{43}(y,x) = -18yw + \sum_{j=1}^{13}\alpha_j u_j', \quad u_j' = u_j(y,x).$$

Now we use $f_{43}(x,y)$ and $f_{43}(y,x)$ to define

$$f_4 = 18x_1w_{33} + \sum_{j=1}^{13}\alpha_j u\rho(u_j), \quad (35)$$

$$f_5 = -18y_1w_{33} + \sum_{j=1}^{13}\alpha_j u\rho(u_j'). \quad (36)$$

The next four equations come from the Cayley–Hamilton theorem (23) and its linearizations (30)

$$\begin{aligned} f_6 &= x_1^3 - \tfrac{1}{2}u_{20}x_1 - \tfrac{1}{3}u_{30}, \\ f_7 &= x_1^2y_1 + x_1y_1x_1 + y_1x_1^2 - u_{11}x_1 - \tfrac{1}{2}u_{20}y_1 - u_{21}, \\ f_8 &= x_1y_1^2 + y_1x_1y_1 + y_1^2x_1 - \tfrac{1}{2}u_{02}x_1 - u_{11}y_1 - u_{12}, \\ f_9 &= y_1^3 - \tfrac{1}{2}u_{02}y_1 - \tfrac{1}{3}u_{03}. \end{aligned} \quad (37)$$

Finally, we define f_{10} and f_{11} using the relations (28) and (29):

$$f_{10} = 6(x_1y_1)^2 - 6y_1^2x_1^2 + 3u_{02}x_1^2 - 6u_{11}x_1y_1 + 3u_{20}y_1^2 + 2\big(-u_{20}u_{02} + u_{11}^2 + u_{22}\big), \quad (38)$$

$$\begin{aligned} f_{11} = {} & 36y_1^2x_1y_1x_1^2 - 6u_{02}x_1y_1x_1^2 + 12u_{11}\big((y_1x_1)^2 - y_1^2x_1^2\big) - 6u_{20}y_1^2x_1y_1 \\ & + 12u_{12}\big(x_1y_1x_1 - y_1x_1^2\big) + 12u_{21}\big(y_1x_1y_1 - y_1^2x_1\big) \\ & + \big(-u_{20}u_{02} + 4u_{11}^2 + 4u_{22}\big)x_1y_1 + 2\big(-u_{20}u_{02} - 2u_{11}^2 + 4u_{22}\big)y_1x_1 \\ & + 2\big(u_{20}u_{03} - 6u_{11}u_{12} + 3u_{02}u_{21}\big)x_1 + 2\big(u_{02}u_{30} - 6u_{11}u_{21} + 3u_{20}u_{12}\big)y_1 \\ & + 2\big(u_{20}u_{11}u_{02} - u_{11}^3 + u_{30}u_{03} - 3u_{21}u_{12} - u_{22}u_{11} - 3w_{33}\big). \end{aligned}$$

The following theorem is the main result of the section. Pay attention that the Gröbner basis which we give is minimal (no leading monomials of its elements are divisible by each other) but is not reduced (some of the summands are not in normal form).

Theorem 3.4. *The Gröbner basis of the kernel of the natural homomorphism $\pi : K[U]\langle x_1, y_1, w_{33}\rangle \to T_{32}$ from* (27) *with respect to the above defined ordering of $[U]\langle x_1, y_1, w_{33}\rangle$ consists of the polynomials f_1–f_{11} from* (33), (34), (35), (36), (37), (38)*, and* (39).

Proof. The leading monomials of f_1–f_{11} are

$$f_1^0 = w_{33}^2, \quad f_2^0 = w_{33}x_1, \quad f_3^0 = w_{33}y_1, \quad f_4^0 = x_1w_{33}, \quad f_5^0 = y_1w_{33},$$
$$f_6^0 = x_1^3, \quad f_7^0 = x_1^2y_1, \quad f_8^0 = x_1y_1^2, \quad f_9^0 = y_1^3,$$
$$f_{10}^0 = (x_1y_1)^2, \quad f_{11}^0 = y_1^2x_1y_1x_1^2.$$

Now we work modulo the ideal $\mathrm{Ker}(\pi)$. Let $a \in [U]$, $f \in \langle x_1, y_1, w_{33}\rangle$ be arbitrary monomials. Using the elements f_1–f_{11} from (33), (34), (35), (36), (37), (38), and (39), we replace af with a linear combination of monomials which are lower in the ordering of $[U]\langle x_1, y_1, w_{33}\rangle$. We continue this process until af is presented as a linear combination of monomials which do not contain any of the leading monomials f_1^0–f_{11}^0, i.e. these monomials are reduced. Hence, without loss of generality we may assume that af is reduced. If f contains w_{33}, then, by (33), (35), and (36), it cannot contain subwords $w_{33}x_1, w_{33}y_1, x_1w_{33}, y_1w_{33}$. By (34) it cannot contain w_{33}^2. Hence, if f contains w_{33}, then $f = w_{33}$. The leading monomials of f_6–f_{11} are those from (31). Now we use Lemma 3.3. If af is reduced, then f may be any monomial from $\langle x_1, y_1\rangle$ of degree $\leqslant 2$ or one of the monomials from (32). The generating function of the set M of all monomials from $\langle x_1, y_1, w_{33}\rangle$ in normal form with respect to $\{f_1, \ldots, f_{11}\}$ (taking into account the degrees of the elements) is

$$\begin{aligned} H(M, t_1, t_2) &= 1 + (t_1 + t_2) + \big(t_1^2 + 2t_1t_2 + t_2^2\big) + 2\big(t_1^2t_2 + t_1t_2^2\big) \\ &\quad + \big(t_1^3t_2 + 2t_1^2t_2^2 + t_1t_2^3\big) + \big(t_1^3t_2^2 + t_1^2t_2^3\big) + t_1^3t_2^3 \\ &= 1 + S_{(1,0)} + (S_{(2,0)} + S_{(1,1)}) \\ &\quad + 2S_{(2,1)} + (S_{(3,1)} + S_{(2,2)}) + S_{(3,2)} + S_{(3,3)}, \end{aligned}$$

and this is the polynomial $p(t_1,t_2)$ from (15). Hence, the Hilbert series of the free $K[U]$-module with basis M is

$$H\big(K[U]M,t_1,t_2\big)=H\big(K[U],t_1,t_2\big)H(M,t_1,t_2).$$

Since the Hilbert series of $K[U]$ is

$$H\big(K[U],t_1,t_2\big)=\frac{1}{(1-t_1)(1-t_2)q_2(t_1,t_2)q_3(t_1,t_2)(1-t_1^2t_2^2)},$$

where q_2,q_3 are defined in (12), by (13) we obtain that

$$H(T_{32},t_1,t_2)=H\big(K[U]M,t_1,t_2\big).$$

As a graded vector space, T_{32} is a homomorphic image of $K[U]M$ and the coincidence of the Hilbert series shows that $T_{32}\cong K[U]M$, i.e. the K-algebra $K[U]\langle x_1,y_1,w_{33}\rangle/\operatorname{Ker}(\pi)$ has a basis consisting of all normal monomials. This implies that $\{f_1,\dots,f_{11}\}$ is a Gröbner basis of $\operatorname{Ker}(\pi)$. $\square$

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